PMM U.S.S.R.,Vol.45,pp.437-444 0021-8928/82/4 0437 $7.50/0

UDC 531.36

ON THE STABILITY OF THE EQUILIBRIUM POSITION IN CRITICAL AND NEAR-CRITICAL CASES*

L.G. KHAZIN and E.E. SHNOL'

Autonomous systems of ordinary differential equations are examined. The stability of the equilibrium position is studied in nearly degenerate (critical) cases. Estimates are given for the magnitude of the small attracting domain under stability in the critical case and for the magnitude of the "dangerous" perturbations under instability. Standard theorems are formulated and examples given of the proof of appropriate theorems. A list is given of all critical cases of degeneration degree no higher than three and of the corresponding stability criteria.

1. Critical cases and their degeneration degree. We consider a real system with smooth right-hand sides

$$du / dt = f(u), \quad f(u^{(0)}) = 0 \qquad (1.1)$$
$$u = (u_1, \ldots, u_n), \quad f = (f_1, \ldots, f_n)$$

Let $(df/du)_{u=u^{(0)}} = \Gamma$ be a linearization matrix and λ_k be its eigenvalues. In the equilibrium position stability problem we distinguish the common case, when the question is resolved in the linear approximation, and the critical cases, in which the answer depends upon the nonlinear terms. We characterize a critical case by a collection of ν equality conditions imposed on the right-hand sides of system (1.1). The conditions can be imposed not only on Γ but also on the higher derivatives of f (allowing for only the conditions affecting the stability). The number ν is called the degeneration degree or the codimension of the problem being examined. As ν increases the difficulty of the investigation increases, in general, and the problem can even be unsolvable (in the sense indicated in /1—3/). In a general stability problem it is natural, it seems, to have a complete investigation of all critical cases of up to some degeneration degree $\nu = \nu_{max}$. However, among cases with $\nu > \nu_{max}$ it is appropriate to examine those stemming from applied problems or those of particular mathematical interest. If we take such a stand, then we have to restrict ourselves to $\nu_{max} = 3$ since at present we cannot set up even a complete list of critical cases with $\nu = 4$. Below, the statement "the system is stable" signifies the asymptotic stability of the equilibrium position $u^{(0)}$, "instability" signifies the absence of Liapunov-stability, n_0 is the number of λ_k equal to zero, and n_2 is the number of pairs of purely imaginary eigenvalues of Γ; $\operatorname{Re}\lambda_j < 0$ for the remaining λ_j.

Example. In system (1.1) let $n_0 = 2, n_2 = 0$, and let there be no additional degeneration $(\nu = 2)$. Then the Jordan normal form of Γ contains a cell. Stability criterion: instability obtains when $a \neq 0$. Here a is some combination of quadratic coefficients. Both stability and instability are possible when $a = 0\ (\nu = 3)$. The stability criterion consists of inequalities on the Taylor coefficients of f up to fourth degree, inclusive. Equating the left-hand side of one of these inequalities to zero, we obtain a problem with $\nu = 4$, not analyzed until now.

2. Statements of the standard theorems. We reduce system (1.1) to a normal form of up to some order m (see /4,5/) and we discard the higher-order terms. We obtain

$$v^{\cdot} = h(v), \quad w^{\cdot} = g(v, w)$$

Here the variables v correspond to critical λ_j $(\operatorname{Re}\lambda_j = 0)$, w to the rest. The polynomial system (M): $v^{\cdot} = h(v)$ which splits off is termed model; its real dimension equals $n_0 + 2n_2$. The investigation of the critical case consists in proving the following theorems.

Theorem M (stability criterion for the model system). For the asymptotic stability of system (M) it is sufficient to fulfil conditions M_+ of the form $\Phi_1(a) < 0, \ldots, \Phi_N(a) < 0$. Here a are the coefficients of system (M). If $\Phi_j(a) \neq 0$ for all j and $\Phi_j(a) > 0$ for even one j, then the equilibrium position $v = 0$ is unstable. In this case we speak of strict instability.

---
*Prikl.Matem.Mekhan.,45,No.4,595-604,1981

Theorem S. If conditions M_+ are fulfilled, system (1.1) is stable.

Theorem NS. If system (M) is strictly unstable, system (1.1) is unstable.

The correctness of the choice of the model system (of the number m) can be established only by Theorems S and NS. Sometimes certain terms of degree no higher than m do not affect the stability and are not included in the model system. Let us now consider systems that differ slightly from (1.1) in the sense C_1:

$$du/dt = f^{(\varepsilon)}(u), \quad \| f^{(\varepsilon)} - f \|_1 < \varepsilon, \qquad \| g \|_1 = \sum_k \max | g_k | + \sum_{k,j} \max \left| \frac{\partial g_k}{\partial x_j} \right| \tag{2.1}$$

Theorem S_ε. Let conditions M_+ be fulfilled for system (1.1). Then, for all systems (2.1) with $\varepsilon \leqslant \varepsilon_0$: 1) a neighborhood G_1 of point $u^{(0)}$ exists such that every trajectory starting in G_1 remains in G_1 for $0 \leqslant t < \infty$; 2) all trajectories starting in G_1 go into a small neighborhood $G_{2\varepsilon}$ of point $u^{(0)}$ as $t \to \infty$. The diameter $d(G_{2\varepsilon}) \leqslant C\varepsilon^{\varkappa_+}$.

Theorem NS_ε. For system (1.1) let the model system be strictly unstable. Then when $\varepsilon \leqslant \varepsilon_0$ there exists a solution $u_\varepsilon(t)$ for each of the systems (2.1), for which $|u_\varepsilon(0)| = \delta(\varepsilon)$, $\sup_{t>0} |u_\varepsilon(t)| > a$. Here $\delta(\varepsilon) \leqslant C\varepsilon^{\varkappa_-}$, but does not depend on ε.

The indices $\varkappa_+$ and $\varkappa_-$ for all cases with $\nu \leqslant 3$ are given in Sect.5.

Notes. 1°. In a number of critical cases (usually when $n_0 \neq 0$) instability is the norm. Theorem M then has the form: "if $\Phi(a) \neq 0$, instability obtains" (see example in Sect.1). In these cases Theorem S and S_ε are absent.

2°. In Theorems S_ε the regular estimate of the diameter $d(G_{2\varepsilon})$ of the attracting domain is a principal one. The proofs use Liapunov functions constructed in accordance with Theorems S. In Theorems NS_ε a regular estimate of $\delta(\varepsilon)$ is most essential. The proof of the fact that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ for system (2.1) if system (1.1) is unstable is elementary (see /6/).

3°. The perturbation $f^{(\varepsilon)} - f$ may depend on t or may be a random function. If only the quantity

$$\| f^{(\varepsilon)} - f \|_0 = \sum_k \max | f_k^{(\varepsilon)} - f_k |$$

is small, then Theorems S_ε and NS_ε remain in force, but the indices $\varkappa_+$ and $\varkappa_-$ may be decreased (see Sect.6).

4°. If matrix Γ is diagonalizable and system (M) contains only mth-degree terms in addition to the linear ones, then an explicit use of conditions M_+ is not required. Then Theorems S and S_ε start thus: "if system (M) is asymptotically stable, then..." (see Sect.6).

5°. For parameter-dependent systems Theorems S_ε and NS_ε are connected with the concepts of soft and hard loss of stability /7/, of dangerous and safe boundaries of the stability domain in parameter space (see /6.8/) (*).

3. List of critical cases with $\nu \leqslant 3$. In order to enumerate all the critical cases with $\nu = 3$ it is necessary to write out the stability criteria in all cases with $\nu = 2$ and to replace by turns one of the inequalities obtained by an equality. Therefore, here we derive the model systems and the stability criteria for $\nu = 1, 2$ (see Sect.4 for the criteria for $\nu = 3$). Below x_k are real variables, $z_k = \sqrt{\rho_k} e^{i\varphi_k}$ are complex variables, the sign «*» denotes the complex conjugate, $a, b, \alpha, \beta, \gamma$ are real numbers, and A, B are complex numbers. The papers in which it seems that the criteria derived were first obtained are mentioned in the references.

We now present the list of critical cases with $\nu \leqslant 3$.

Cases 1–3: $n_0 = 1$, $n_2 = 0$ /9/.

1) $(M): x^\cdot = ax^2$. Instability when $a \neq 0$ $(\nu = 1)$.

2) Additional degeneration: $a = 0$, $\nu = 2$. $(M): x^\cdot = bx^3$. Stability when $b < 0$, instability when $b > 0$.

---

*) E.E. Shnol' and L.G. Khazin, On the stability of stationary solutions of general systems of differential equations in near-critical cases. M: Akad. Nauk SSSR Inst. Prikl. Mat. Preprint No.91, 1979; L.G. Khazin and E.E. Shnol', On soft and hard loss of stability of steady-state solutions of differential equations. M.: Akad. Nauk SSSR Inst. Prikl. Mat. Preprint No.128, 1979.

3) Repeated additional degeneration: $a = b = 0$, $\nu = 3$.
Cases 4–6: $n_0 = 0$, $n_2 = 1$ /9/
4) $\nu = 1$; $(M): z^{\cdot} = i\omega z + A_1 z\rho$. Stability when $a_1 < 0$, instability when $a_1 > 0$, $a_1 = \operatorname{Re} A_1$.
5) Additional degeneration: $a_1 = 0, \nu = 2$; $(M)\, z^{\cdot} = i\omega z + z(A_1\rho + A_2\rho^2)$. Stability when $a_2 < 0$, instability when $a_2 > 0$, $a_2 = \operatorname{Re} A_2$.
6) Repeated additional degeneration: $a_1 = a_2 = 0$.
Cases 7–8 : $n_0 = 2$, $n_2 = 0$ /10/.
7) $\nu = 2$; $(M): x_1^{\cdot} = x_2$, $x_2^{\cdot} = a x_1^2$. Instability when $a \neq 0$.
8) Additional degeneration: $a = 0$, $\nu = 3$.

Note 6°. The case when two eigenvectors of Γ are associated with a multiple $\lambda = 0$ corresponds to $\nu = 4$ and, therefore, is absent in this list.
Cases 9–10: $n_0 = n_2 = 1$.
9) $\nu = 2$; $(M): z^{\cdot} = i\omega z$, $x^{\cdot} = ax^2$. Instability when $a \neq 0$ /9/.
10) Additional degeneration: $a = 0, \nu = 3$ (*).
11) $n_0 = 3$, $n_2 = 0$, $\nu = 3$ /11/.
With two pairs of purely imaginary λ $(n_0 = 0,\ n_2 = 2)$ we have one case of "general position" $(\nu = 2)$ and five with additional degeneration: $(\nu = 3)$. In three of them an additional condition (the resonance relation) is imposed on the linear terms, while in the other two, on the nonlinear (quadratic and cubic) ones. Let

$$\lambda_{1,2} = \pm i\omega_1, \quad \lambda_{3,4} = \pm i\omega_2, \quad \omega_2 > \omega_1 > 0, \quad \omega_2 \neq 2\omega_1, \quad \omega_2 \neq 3\omega_1$$

12) $\nu = 2; (M): z_k^{\cdot} = i\omega_k z_k + z_k(A_{k1}\rho_1 + A_{k2}\rho_2), k = 1, 2$ /11/.
$M_+: a_{11} < 0$, $a_{22} < 0$; when $a_{12} > 0$ and $a_{21} > 0\ \Delta = a_{11}a_{22} - a_{12}a_{21} > 0$; $a_{jk} = \operatorname{Re} A_{jk}$.

13) Additional degeneration: $a_{11} = 0$, $\nu = 3$ /6/.
14) Additional degeneration: $\Delta = 0$, $\nu = 3$ when $a_{12} > 0$ and $a_{21} > 0$. Two pairs of purely imaginary λ and internal resonance: cases 15–17.
15) Resonance $1:1\ (\omega_2 = \omega_1)$; $\nu = 3$ (**).
16) Resonance $1:2\ (\omega_2 = 2\omega_1)$; $\nu = 3$ /12,13/.
17) Resonance $1:3\ (\omega_2 = 3\omega_1)$; $\nu = 3$ /3/.
Remaining cases:
18) $n_0 = 1$, $n_2 = 2$, $\nu = 3$.
19) $n_0 = 2$, $n_2 = 1$, $\nu = 3$.
20) $n_0 = 0$, $n_2 = 3$, $\nu = 3$ /14/.

4. Stability criteria for cases of codimension 3.

3) $(M): x^{\cdot} = cx^4$. Instability when $c \neq 0$.
6) $(M): z^{\cdot} = i\omega z + z(A_1\rho + A_2\rho^2 + A_3\rho^3)$, $\operatorname{Re} A_1 = \operatorname{Re} A_2 = 0$. Stability when $\operatorname{Re} A_3 < 0$ and instability when $\operatorname{Re} A_3 > 0$.
8) $(M): x^{\cdot} = y + d_0x^2 + dxy + d_1x^3$; $y^{\cdot} = ax^3 + bxy + a_1x^4 + b_1x^2y + cy^2$. Conditions M_+ are three inequalities $P < 0$, $Q^2 < 8P^2$, $g < 0$. Here

$$P = a - d_0 b;\ Q = b + 2d_0,\ g = (c + 3a_1/a - 2d)\, Q + 5Q_1;\ Q_1 = b_1 - 2d_0c - d_0d + 3d_1.$$

The third and fourth degree terms not written out are unessential for stability (***).
10) $(M): x^{\cdot} = b\,|z|^2 + b_2x^3$; $z^{\cdot} = i\omega z + Azx$. Conditions M_+ have the form: $ab < 0$, $b_2 < 0$; $a = \operatorname{Re} A$.
11) $(M): x_1^{\cdot} = x_2$, $x_2^{\cdot} = x_3$, $x_3^{\cdot} = ax_1^2$. Instability when $a \neq 0$.
13) (M): $\rho_1^{\cdot} = \rho_1(a_{12}\rho_2 + a_{11}^{(1)}\rho_1^2)$; $\rho_2^{\cdot} = \rho_2(a_{21}\rho_1 + a_{22}\rho_2)$; $\psi_k^{\cdot} = h_k(\rho, \varphi)$.
Here $a_{22} < 0$, a_{12} and a_{21} are not simultaneously positive. Condition $M_+ : a_{11}^{(1)} < 0$.

14) $$(M): \begin{cases} \rho_1^{\cdot} = \rho_1(-\rho_1 + a\rho_2) + \rho_1 \sum a_{jk}^{(1)}\rho_j\rho_k \\ \rho_2^{\cdot} = l\rho_2(\rho_1 - a\rho_2) + \rho_2 \sum a_{jk}^{(2)}\rho_j\rho_k, \quad \varphi_k^{\cdot} = h_k(\varphi, \rho) \end{cases}$$

---

*) L.G. Khazin, On the stability of the equilibrium position in some critical cases. M.: Akad. Nauk SSSR Inst. Prikl.Mat.Preprint No.10, 1979.
**) L.G. Khazin, On resonance instability of the equilibrium position under multiple frequencies. M.: Akad. Nauk SSSR Inst. Prikl. Mat. Preprint No.97, 1975.
***) For a discussion of this criterion and its simple derivation see: L.G. Khazin, M.:Akad. Nauk SSSR Inst. Prikl. Mat. Preprint No.9, 1980.

Here $a > 0$, $l > 0$. Condition $M_+ : K < 0$. Here

$$K = a^2(la_{11}^{(1)} + a_{11}^{(2)}) + a(la_{12}^{(1)} + a_{12}^{(2)}) + la_{22}^{(1)} + a_{22}^{(2)}.$$

15) $(M) : z_1^{\cdot} = i\omega z_1 + z_2;\ z_2^{\cdot} = i\omega z_2 + Bz_1 \mid z_1 \mid^2$. Instability when $\operatorname{Im} B \neq 0$.

16) $(M) : z_1^{\cdot} = i\omega z_1 + B_1 z_1^* z_2;\ z_2^{\cdot} = 2i\omega z_2 + B_2 z_1^2$. Instability when $B_1 \neq 0$, $B_2 \neq 0$, $B_2 / B_1^* \neq \gamma < 0$.

17) The stability criterion cannot be specified by explicit formulas.

18) (M): $x^{\cdot} = ax^2$; $z_k^{\cdot} = i\omega_k z_k$, $k = 1, 2$. Instability when $a \neq 0$.

19) (M): $x_1^{\cdot} = x_2$, $x_2^{\cdot} = ax_1^2$, $z^{\cdot} = i\omega z$.

20) Let $\lambda_{1,2} = \pm i\omega_1$, $\lambda_{3,4} = \pm i\omega_2$; $\lambda_{5,6} = \pm i\omega_3$ and let lower-order resonances be absent: $\omega_j \neq \omega_k$, $\omega_j \neq 2\omega_k$, $\omega_j \neq 3\omega_k$, $\omega_j \neq \omega_k + \omega_l$; $\omega_j \neq 2\omega_k + \omega_l$.

$$(M):\ \rho_k^{\cdot} = \rho_k \sum a_{kj}\rho_j;\quad k, j = 1, 2, 3$$

The equations for φ_k are not written out. A solution of form $\rho_k(t) = c_k r(t)$, $r^{\cdot} = r^2$, $c_k \geqslant 0$ (not all $c_k = 0$) is called a growing invariant ray of system (M). Condition M_+: there are no growing invariant rays among the solutions of system (M). The verification of M_+ leads to a consideration of seven linear systems $A\rho = e$ with $\rho_k \geqslant 0$. Here $A = \| a_{jk} \|$ and as e we take the vectors $(1, 1, 1), \ldots, (0, 0, 1)$. If even one of these systems has a solution, then system (M) is unstable.

5. Indices in systems S_ε and NS_ε. Table 1 shows the indices $\varkappa_+$ and $\varkappa_-$ figuring in Theorems S_ε and NS_ε. The majority of the indices presented are unimprovable. If the indices can be increased, then the hypothetical unimproved value has been shown within parentheses. We observe that if the requirement $u^{(0)} \in G_{2\varepsilon}$ is waived, then the indices $\varkappa_+$ can be increased in certain cases where there are zero λ. A dash in the $\varkappa_+$ column signifies that stability is impossible in the case given. In cases 8 and 10 the indices $\varkappa_-$ depend upon which of the conditions are violated.

Table 1

N	$\varkappa_+$	$\varkappa_-$	N	$\varkappa_+$	$\varkappa_-$
1	—	$1/2$	11	—	$5/6$
2	$1/3$	$1/2$	12	$1/2$	$1/2$
3	—	$1/4$	13	$1/4$	$1/4$
4	$1/2$	$1/2$	14	$1/4$	$1/4$ ($1/4$)
5	$1/4$	$1/4$	15	—	1
6	$1/6$	$1/6$	16	—	1
7	—	$3/4$	17	$1/2$	$1/2$
8	$1/4$	$1/3$; $1/4$	18	·	$1/2$
9	—	$1/2$	19	—	$1/2$ ($3/4$)
10	$1/3$	$1/2$; $1/3$ ($1/2$)	20	$1/2$	$1/2$

6. Example of the proof of Theorem S_ε (case 20). Let system (1.1) satisfy the conditions of case 20 (Sect.3). By smooth transformation (invertible in a neighborhood of $u^{(0)}$) we normalize (1.1) up to third-order terms, inclusive. We obtain

$$z_k^{\cdot} = i\omega_k z_k + p_k^{(3)}(z) + r_{1k};\quad w_j^{\cdot} = \lambda_j w_j + q_j^{(2)}(w) + q_j^{(3)}(z, w) + r_{2j} \tag{6.1}$$

Here z_k are complex variables ($k = 1, 2, 3$), w_j are complex or real variables. If there are multiple λ_j, then we can also take linear summands in the equations for w. The upper index gives the degree of the homogeneous polynomial; polynomials $q_j^{(3)}$ are linear in the variable w; $| r_{1k}(z, w) |, | r_{2j}(z, w) | \leqslant C(| z |^4 + | w |^4)$; for the vectors we have

$$| z |^2 = \sum_k | z_k |^2,\ | w |^2 = \sum_j | w_j |^2 \ldots.$$

In a real notation system (6.1) takes the form

$$\begin{aligned} &x^{\cdot} = \Gamma_1 x + P^{(3)}(x) + R^{(x)} \\ &y^{\cdot} = \Gamma_2 y + Q^{(2)}(y) + Q^{(3)}(x, y) + R^{(y)} \\ &| Q^{(3)} | \leqslant C | x |^2 y,\quad | R^{(x)}(x, y) | \\ &| R^{(y)}(x, y) | \leqslant C(| x |^4 + | y |^4) \\ &\dim x = 6,\quad \dim y - n - 6 \end{aligned} \tag{6.2}$$

In the real notation the system

$$\dot{x} = \Gamma_1 x + P^{(3)}(x) \tag{6.3}$$

corresponds to the model system (M). Let us consider an arbitrary system of form (6.3), being the real notation for a normalized system. Let Γ_1 be diagonalizable and all $\operatorname{Re}\lambda(\Gamma_1) = 0$.

Lemma 1 /14/. Let $x = \Phi_1(t, \xi)$ be a general solution of system (6.3) and $x = \Phi_2(t, \xi)$ be a general solution of the homogeneous system $x^{\cdot} = P^{(3)}(x)$ $(\Phi_k \times (0, \xi) = \xi)$. Then $\Phi_1(t, \xi) = \exp(t\Gamma_1)\Phi_2(t, \xi)$; in particular, $|\Phi_1(t, \xi)| = |\Phi_2(t, \xi)|$.

Lemma 2. Let the homogeneous system

$$x^{\cdot} = h(x), \quad h(\alpha x) = \alpha^m h(x),\ m > 1 \quad (\forall \alpha > 0) \tag{6.4}$$

be given. If it is asymptotically stable, then a homogeneous Liapunov function exists for which

$$L^{\cdot} \leqslant -\gamma |x|^{m+1},\ L(\alpha x) = \alpha^2 L(x)\ (\gamma > 0) \tag{6.5}$$

Corollary. An $\varepsilon_0 > 0$ exists such that when $|\delta h(x)| < \varepsilon_0 |x|^m$ the system $x^{\cdot} = h(x) + \delta h(x)$ is asymptotically stable simultaneously with system (6.4).

Note 7°. Lemma 2 is a corollary of Theorem 22.1 in /15/ (the appropriate references are presented therein).

Corollary. From Lemmas 1 and 2 it follows that if system (6.3) is stable, it admits of a homogeneous Liapunov function $L_1(x)$ with an estimation (6.5).

Lemma 3 (Theorem S). If system (6.3) is stable, so is system (6.2).

Proof. Let $L_2(y)$ be a quadratic Liapunov function for the system $y^{\cdot} = \Gamma_2 y + Q^{(2)}(y)$, with the estimation $L_2^{\cdot} < -\gamma_2 |y|^2$. On the strength of the homogeneity, $|\partial L_1/\partial x| \leqslant C_1 |x|, |\partial L_2/\partial y| \leqslant C_1 |y|$. We set

$$L(u) = L(x, y) = L_1(x) + L_2(y)$$
$$u = (u_1, \ldots, u_n) = (x_1, \ldots, x_6, y_1, \ldots, y_{n-6})$$
$$L^{\cdot}|_{(6.2)} \leqslant -\gamma_1 |x|^4 + C_2 |x|\,|u|^4 - \gamma_2 |y|^2 + C_2 |y|\,(|x|^2 |y| + |u|^4) \leqslant -\gamma_3 |u|^4 + C_3 |u|^5,\ \gamma_3 > 0$$

By virtue of (6.2)

$$L^{\cdot} < -\gamma |u|^4, \quad (|u| < C_0,\ \gamma > 0) \tag{6.6}$$

Theorem 1 (Theorem S for case 20). If system (6.3) is stable, then... (this continues in accord with the standard formulation in Sect.2), and the index $\varkappa_+ = 1/2$.

Proof. By U we denote the domain of action of inequality (6.6), by $U(a)$ the domain $0 \leqslant L \leqslant a$, and by $S(a)$, the boundary of $U(a)$. Since $L(\alpha u) = \alpha^2 L(u)$,

$$K_1 a^{1/2} \leqslant |u| \leqslant K_2 a^{1/2} \tag{6.7}$$

when $u \in S(a)$. Let $U(a) \subset U$ when $a < a_1$. We set $G_1 = U(a_1)$. When $\varepsilon = 0$, $L^{\cdot} < \eta < 0$ on $S(a_1)$. Therefore, when $\varepsilon < \varepsilon_1$, because of the perturbed system (2.1), $L^{\cdot} < \eta/2$ on $S(a_1)$, i.e., the trajectories of any one of systems (2.1) goes into domain G_1. The derivative of L relative to system (2.1) is

$$L^{\cdot} = \sum \frac{\partial L}{\partial u_k}(f_k + f_k^{(\varepsilon)} - f_k) \leqslant -\gamma |u|^4 + C_1 \varepsilon |u| \tag{6.8}$$

The right-hand side of (6.8) is negative when $|u| > C_2 \varepsilon^{1/3}$, i.e., by reason of (2.1) $u(t)$ "monotonically" (in the sense of decrease of L) approaches $u = 0$ up to a distance $\sim \varepsilon^{1/3}$. We remark that in (6.8) we made use only of the smallness of $|f - f^{(\varepsilon)}|$. Let $\|f^{(\varepsilon)} - f\|_1 < \varepsilon$. When $\varepsilon < \varepsilon_0$, by the implicit function theorem there exists, in a neighborhood of $u = 0$, a unique solution $u^{(\varepsilon)}$ of the system $f^{(\varepsilon)}(u) = 0$; in this connection, $|u^{(\varepsilon)}| \leqslant C\varepsilon$. We denote $\delta f(u) = f^{(\varepsilon)}(u) - f(u - u^{(\varepsilon)})$. Then

$$\left|\frac{\partial(\delta f)}{\partial u_k}\right| \leqslant \left|\frac{\partial f^{(\varepsilon)}(u)}{\partial u_k} - \frac{\partial f(u)}{\partial u_k}\right| + \left|\frac{\partial f(u)}{\partial u_k} - \frac{\partial f(u - u^{(\varepsilon)})}{\partial u_k}\right|$$

The first summand $< \varepsilon$ by (2.1) and the second $< C_3 \varepsilon$ because f is smooth. Since $\delta f(u^{(\varepsilon)}) = 0$, we have

$$|\delta f(u)| < C_4 \varepsilon\, |u - u^{(\varepsilon)}| \tag{6.9}$$

We fix some function $f^{(\varepsilon)}$; let $L^{(\varepsilon)}(u) = L(u - u^{(\varepsilon)})$. The derivative of $L^{(\varepsilon)}$ relative to (2.1) has the form

$$\frac{dL^{(\varepsilon)}}{dt} \equiv \frac{\partial L^{(\varepsilon)}}{\partial u} f^{(\varepsilon)}(u) = \frac{\partial L}{\partial u}(u - u^{(\varepsilon)})(f^{(\varepsilon)}(u) - f(u - u^{(\varepsilon)})) + \frac{\partial L}{\partial u}(u - u^{(\varepsilon)}) f(u - u^{(\varepsilon)}) \leqslant C_5\varepsilon|u - u^{(\varepsilon)}|^2 - \gamma|u - u^{(\varepsilon)}|^4 \quad (6.10)$$

Here the first summand has been estimated on the strength of (6.9) and the second, of (6.6). The right-hand side of (6.10) is less than $-|u - u^{(\varepsilon)}|^4\gamma/2$ when $|u - u^{(\varepsilon)}| > C_6\varepsilon^{1/2}$. We set $G_2 = \{u, L^{(\varepsilon)}(u) < q\varepsilon\}$, $q = (C_6/K_1)^2$. A neighborhood of G_2 is obtained from $U(q\varepsilon)$ by a shift by an amount $\leqslant C\varepsilon$ and is invariant for the system (2.1) selected. When $\varepsilon \leqslant \varepsilon_2$ the domain $U(2q\varepsilon)$ contains any of the domains G_2; we set $G_{2\varepsilon} = U(2q\varepsilon)$. Then the diameter $d(G_{2\varepsilon}) \leqslant K_2(2q\varepsilon)^{1/2} = C_7\varepsilon^{1/2}$ (see (6.7) for K_1 and K_2).

Notes. 8°. In the proof of Theorem 1 we have used only: a) the asymptotic stability of the model system; b) the absence of quadratic terms in (6.3). Therefore, the theorem proved is valid for any $n_2 \geqslant 1$ in the absence of resonances of the two lowest orders ($\omega_j \neq \omega_l$; $\omega_j \neq 2\omega_l$, $\omega_j \neq \omega_l + \omega_k$).

9°. If we assume the smallness of only $|f^{(\varepsilon)} - f|$, then for any $n_2 \geqslant 1$ the unimprovable index $\varkappa_+$ is 1/3. Let us clarify this by the example of $n_2 = 1$. Let a system of two equations in the complex notation be

$$z' = i\omega z + \varepsilon^{2/3} h(\rho\varepsilon^{-2/3}) z - \rho z \quad (\rho = |z|^2) \qquad (6.11)$$

Here $h(\rho) \equiv 1$ when $\rho \leqslant 1$; $0 \leqslant h(\rho) < 1$ when $1 < \rho < 2$, $h(\rho) \equiv 0$ when $\rho \geqslant 2$. Among the solutions of (6.11) there is the stable limit cycle $|z| = \varepsilon^{1/3}$ and $G_{2\varepsilon} = \{z, |z| \leqslant \varepsilon^{1/3}\}$. Then $|f - f^{(\varepsilon)}| \leqslant \varepsilon$, $\|f - f^{(\varepsilon)}\|_1 \sim \varepsilon^{1/3}$.

7. Example of the proof of Theorem NS_ε (case 16).

The proof scheme suggested below is typical. The model system's instability can be proved by constructing a solution whose trajectory l tends to the equilibrium position as $t \to -\infty$. We construct a canonic neighborhood $\Omega(l)$ with the following properties: a) the solutions of the complete system grow monotonically in some metric inside Ω; b) the trajectories only enter Ω through its lateral boundary Ω (compare with /16,17/). When $\varepsilon \neq 0$ properties a) and b) are preserved in a neighborhood of $u^{(0)}$ of the size of the order of $\varepsilon^{\varkappa_-}$, which yields Theorem NS_ε. In certain cases we have managed to construct polynomial Chetaev functions in a complete neighborhood of $u^{(0)}$ (or in the greater part of it); this permits us to construct the proof and to obtain unimprovable indices $\varkappa_-$ is cases 7 and 15 and some others. This scheme is not discussed here.

Let system (1.1) satisfy the conditions of case 16. Let dimension $n = 4$ (i.e., the smallest value possible for this case). Lemmas 4 and 5 repeat the results of /12,13/ in a form in which they are to be used in what follows. We normalize system (1.1) up to second-order terms, inclusive. In complex form we obtain

$$z_1' = i\omega z_1 + B_1 z_1^* z_2 + O(|z|^3), \quad z_2' = 2i\omega z_2 + B_2 z_1^2 + O(|z|^3) \qquad (7.1)$$

The model system is

$$z_1' = i\omega z_1 + B_1 z_1^* z_2, \quad z_2' = 2i\omega z_2 + B_2 z_1^2 \qquad (7.2)$$

We set

$$z_k = \sqrt{\rho_k}\, e^{i\varphi_k}, \quad B_1 = b_1 e^{-i\beta_1}, \quad B_2 = b_2 e^{i\beta_2}, \ \psi = \varphi_2 - 2\varphi_1, \ k = 1, 2$$

From (7.2) we obtain

$$\begin{aligned} \rho_1' &= 2b_1\rho_1\rho_2^{1/2}\cos(\psi - \beta_1) \\ \rho_2' &= 2b_2\rho_1\rho_2^{1/2}\cos(\psi - \beta_2) \\ \psi' &= -2b_1\rho_1^{1/2}\sin(\psi - \beta_1) - b_2\rho_1\rho_2^{-1/2}\sin(\psi - \beta_2) \end{aligned} \qquad (7.3)$$

Lemma 4. Let $b_1 > 0$, $b_2 > 0$, $\beta = \beta_1 - \beta_2 \neq \pi$. Then the equilibrium position (0, 0) of system (7.2) is unstable.

Proof. $\rho_2 = r(t)$, $\rho_1 = kr(t)$, $r' = 2b_1 k^{1/2}\cos\psi_0$, $\psi = \psi_0 = \text{const}$ is a growing solution of system (7.3) if when $\beta \neq 0$ we set

$$k = \frac{b_1}{2b_2}[(8 + \cos^2\beta)^{1/2} - \cos\beta] \qquad (7.4)$$

$$\operatorname{tg}\psi_0 = \frac{b_2 k - b_1\cos\beta}{b_1\sin\beta}, \quad \cos\psi_0 > 0$$

When $\beta = 0$; $k = b_1/b_2$, $\psi_0 = 0$.

Lemma 5. (Theorem NS). Let $b_1 > 0$, $b_2 > 0$, $\beta \neq \pi$. Then the equilibrium position of system (7.1) is unstable.

Proof. We set $\rho_1 = R\cos\theta$, $\rho_2 = R\sin\theta$, $d\tau = R^{1/2}dt$. From (7.1) we obtain

$$R^{\cdot} = R\Pi(\theta,\psi) + O(R^{3/2}) \quad ({}^{\cdot} \equiv d/d\tau) \tag{7.5}$$

$$\Pi = \cos\theta(\sin\theta)^{1/2}[b_1\cos\theta\cos(\psi-\beta_1) + b_2\sin\theta\cos(\psi-\beta_2)]$$

$$\theta^{\cdot} = g_1(\theta,\psi) + O(R^{1/2}),\ \psi^{\cdot} = g_2(\theta,\psi) + O(R^{1/2}) \tag{7.6}$$

$$g_1 = \cos\theta(\sin\theta)^{1/2}[-b_1\sin\theta\cos(\psi-\beta_1) + b_2\cos\theta\cos(\psi-\beta_2)]$$

$$g_2 = (\sin\theta)^{-1/2}[2b_1\sin\theta\sin(\psi-\beta_1) + b_2\cos\theta\sin(\psi-\beta_2)]$$

To a growing solution of the model system corresponds a fixed point $P(\theta_0, \psi_0)$ of the angular system

$$\theta^{\cdot} = g_1(\theta,\psi),\ \psi^{\cdot} = g_2(\theta,\psi) \tag{7.7}$$

In this connection, $\operatorname{ctg}\theta_0 = k$, $\Pi(\theta_0,\psi_0) = \Pi_0 > 0$. Linearizing system (7.7) in P, we obtain the matrix

$$\Lambda = \frac{d(g_1,g_2)}{d(\theta,\psi)} = kb_2\begin{Vmatrix} -3\cos\psi_0 & -\sin\psi_0 \\ 3\sin\psi_0 & -2\cos\psi_0 \end{Vmatrix}$$

Since $\cos\psi_0 > 0$, both the eigenvalues of matrix Λ lie in the left halfplane. Let $l(\theta,\psi)$ be a quadratic Liapunov function such that relative to (7.7) the derivative $l^{\cdot} \leqslant -\gamma l$ in a neighborhood $U_1(P)$. Let a neighborhood $U_2(P) \subset U_1$ be such that on the strength of (7.5) we have $R^{\cdot} > {}^1/_2\Pi_0 R$ when $(\theta,\psi) \in U_2$, $R < R_1$. Let $l_* > 0$ be such that the line $l(\theta,\psi) = l_*$ lies wholly in U_2. Finally, let $R_2 < R_1$ be such that on the strength of (7.6) we have $l^{\cdot} < -\gamma_1 < 0$ when $l(\theta,\psi) = l_*$ and $R < R_2$. Every solution of systems (7.5) and (7.6) (and, by the same token, of system (7.1)) for which $l(\theta(0),\psi(0)) < l_*$, $R(0) = \delta < R_2$ when $\tau = 0$ grows monotonically $(R^{\cdot} > 0)$ up to the fulfillment of the equality $R = R_2$. The lemma has been proved. Now, together with the system $z^{\cdot} = F(z)$ of (7.1), we consider the auxiliary system $z^{\cdot} = F^{(\varepsilon)}(z)$, $z = (z_1, z_2)$ or (in the real notation) the system (2.1).

Theorem 2 (Theorem NS_ε for case 16). The theorem's statement is the standard one (see Sect.2), the index $\varkappa_- = 1$.

Proof. From the implicit function theorem it follows that system $F^{(\varepsilon)}(z) = 0$ has when $\varepsilon < \varepsilon_0$ a unique solution $z^{(\varepsilon)}$ close to $z = 0$: $F^{(\varepsilon)}(z^{(\varepsilon)}) = 0$, $|z^{(\varepsilon)}| < C\varepsilon$. In this connection (see Sect. 6) $F^{(\varepsilon)}(z) = F(z - z^{(\varepsilon)}) + \delta F$; $|\delta F| \leqslant C_1\varepsilon|z - z^{(\varepsilon)}|$. We introduce the variable $v = z - z^{(\varepsilon)}$ i.e. $v_1 = z_1 - z_1^{(\varepsilon)}$, $v_2 = z_2 - z_2^{(\varepsilon)}$). We set

$$v_k = \sqrt{\rho_k}\,e^{i\varphi_k},\quad \psi = \varphi_2 - 2\varphi_1,\quad \rho_1 = R\cos\theta,\quad \rho_2 = R\sin\theta,\ d\tau = R^{1/2}dt \tag{7.8}$$

From the system $z^{\cdot} = E^{(\varepsilon)}(z)$ we obtain

$$R^{\cdot} = R\Pi(\theta,\psi) + O(R^{3/2}) + \Delta\Pi(R,\theta,\psi),\quad |\Delta\Pi| \leqslant CR^{1/2}\varepsilon \tag{7.9}$$

$$\theta^{\cdot} = g_1(\theta,\psi) + O(\sqrt{R}) + \Delta g_1$$

$$\psi^{\cdot} = g_2(\theta,\psi) + O(\sqrt{R}) + \Delta g_2,\quad |\Delta g| < C\varepsilon R^{-1/2}$$

Let $l(\theta,\psi) < l_*$, $R < R_2$ (see Lemma 5). Then, by virtue of (7.9), $R^{\cdot} > {}^1/_2\Pi_0 R - CR^{1/2}\varepsilon$, which exceed ${}^1/_4\Pi_0 R$ when $R > C_1\varepsilon^2$. Let $l(\theta,\psi) = l_*$. Then when $R < R_2$, by virtue of (7.9) we have $l^{\cdot} \leqslant -\gamma_1 + C_2\varepsilon R^{-1/2}$, which is less than $-\gamma_1/2$ when $R > C_3\varepsilon^2$. If $R(0) > C_0\varepsilon^2$ $(C_0 = \max(C_1, C_3))$ $l(\theta(0), \psi(0)) < l_*$, then $R(\tau)$ grows monotonically until $R = R_2$ is achieved. We note that ${}^1/_2|v|^2 < R < |v|^2$. Thus, there exist $v(\tau)$ for which $|v(0)| \leqslant C_4\varepsilon$ and $\sup_{\tau>0}|v(\tau)| > R_2^{1/2}$. For $z = v + z^{(\varepsilon)}$ we obtain $|z(0)| \leqslant C_5\varepsilon$ and $\sup_{t>0}|z(t)| \geqslant a = {}^1/_2 R_2^{1/2}$.

The authors thank the participants of the university program "Methods of investigating the Steady-State Motions of Mechanical Systems" for constructive criticism. The authors are grateful to V.V. Beletskii, V. Sh. Burd, I. K., Gavrilov, Iu.S. Kolesov and V.I. Iudovich for useful discussions.

REFERENCES

1. ARNOL'D V.I., Algebraic unsolvability of the Liapunov-stability problem and of the problem of topological classification of the singular points in an analytic system of differential equations. FUNKTSIONAL'NYII ANALIZ, Vol.4, No.3, 1970.
2. IL'IASHENKO Iu.S., Analytic unsolvability of stability problem and the problems of topological classification of the singular points in analytic systems of differential equations. Mat. Sb., Vol.99, No.2, 1972.

3. KHAZIN L.G. and SHNOL' E.E., Elementary cases of algebraic unsolvability in asymptotic stability problems. Dokl. Akad. Nauk SSSR, Vol.240, No.6, 1978.
4. ARNOL'D V.I., Further Chapters in the Theory of Ordinary Differential Equations. Moscow, NAUKA, 1978.
5. BRIUNO A.D., A Local Method of Nonlinear Analysis of Differential Equations. Moscow, NAUKA, 1979.
6. MALKIN I.G., Theory of Stability of Motion. Moscow, NAUKA, 1966.
7. ANDRONOV A.A., VITT A.A. and CHAIKIN S.E., Theory of Oscillations. (See also in English Pergamon Press, Book No. 09981, U.S.A., Addison-Wesley, 1966).
8. BAUTIN N.N., Behavior of Dynamic Systems Near the Boundary of the Stability Domain. Leningrad—Moscow, GOSTEKHIZDAT, 1949.
9. LIAPUNOV A.M., General Problem of the Stability of Motion. In: Collected Works, Vol.2. Moscow—Leningrad, Izd. Akad. Nauk SSSR, 1956.
10. LIAPUNOV A.M., Investigation of one of the singular cases of the motion stability problem. In: Collected Works, Vol.2, Moscow—Leningrad, Izd. Akad. Nauk SSSR, 1956.
11. KAMENKOV G.V., Stability and Oscillations of Nonlinear Systems. In: Selected Works, Vol.2. Moscow, NAUKA, 1972.
12. KUNITSYN A.L., On stability in the critical case of pure imaginary roots under internal resonance. Differents. Uravnen., Vol.7, No.9, 1971.
13. KHAZINA G.G., Certain stability questions in the presence of resonances. PMM Vol.38, No.1. 1974.
14. MOLCHANOV A.M., Stability in the case of neutrality of the linear approximation. Dokl. Akad. Nauk SSSR, Vol.141, No.1, 1961.
15. KRASOVSKII N.N., Some Problems in the Theory of Stability of Motion. Moscow, FIZMATGIZ, 1959.
16. CHETAEV N.G., One instability theorem. Dokl. Akad. Nauk SSSR, Vol.1, No.9, 1934.
17. ROUCHE N., HABETS P. and LALOY M., Stability Theory by Liapunov's Direct Method (Applied Mathematical Sciences 22). New York—Heidelberg—Berlin, Springer-Verlag, 1977.

Translated by N.H.C.